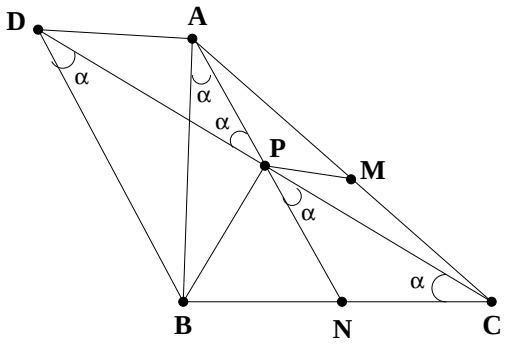
24th Indian National Mathematical Olympiad, 2009

Problems and Solutions

1. Let ABC be a triangle and let P be an interior point such that $\angle BPC = 90^\circ$, $\angle BAP = \angle BCP$. Let M, N be the mid-points of AC, BC respectively. Suppose $BP = 2PM$. Prove that A, P, N are collinear.

Solution:
Extend CP to D such that $CP = PD$. Let $\angle BCP = \alpha = \angle BAP$. Observe that BP is the perpendicular bisector of CD. Hence $BC = BD$ and BCD is an isosceles triangle. Thus $\angle BDP = \alpha$. But then $\angle BDP = \alpha = \angle BAP$. This implies that B, P, A, D all lie on a circle. In turn, we conclude that $\angle DAB = \angle DPB = 90^\circ$. Since P is the mid-point of CP(by construction) and M is the mid-point of CA(given), it follows that PM is parallel to DA and $DA = 2PM = BP$. Thus $DBPA$ is an isosceles trapezium and DB is parallel to PA.



We hence get

$$\angle DPA = \angle BAP = \angle BCP = \angle NPC;$$

the last equality follows from the fact that $\angle BPC = 90^\circ$, and N is the mid-point of CB so that $NP = NC = NB$ for the right-angled triangle BPC. It follows that A, P, N are collinear.

Alternate Solution:

We use coordinate geometry. Let us take $P = (0, 0)$, and the coordinate axes along PC and PB; We take $C = (c, 0)$ and $B = (0, b)$. Let $A = (u, v)$. We see that $N = (c/2, b/2)$ and $M = \big((u + c)/2, v/2\big)$. The condition $PB = 2PM$ translates to

$$(u + c)^2 + v^2 = b^2.$$

We observe that the slope of $CP = 0$; that of CB is $-b/c$; that of PA is v/u; and that of BA is $(v - b)/u$. Taking proper signs, we can convert $\angle PCB = \angle PAB$, via tan function, to the following relation:

$$u^2 + v^2 - vb = -cu.$$

Thus we obtain

$$u(u + c) = v(b - v), \quad c(c + u) = b(b - v).$$

It follows that $v/u = b/c$. But then we get that the slope of AP and PN are the same. We conclude that A, P, N are collinear.

2. Define a sequence $\langle a_n \rangle_{n=1}^{\infty}$ as follows:

$$a_n = \begin{cases} 0, & \text{if the number of positive divisors of } n \text{ is } \textit{odd}, \\ 1, & \text{if the number of positive divisors of } n \text{ is } \textit{even}. \end{cases}$$

(The positive divisors of n include 1 as well as n.) Let $x = 0.a_1a_2a_3\ldots$ be the real number whose decimal expansion contains a_n in the n-th place, $n \geq 1$. Determine, with proof, whether x is rational or irrational.

Solution:

We show that x is irrational. Suppose that x is rational. Then the sequence $\langle a_n\rangle_{n=1}^{\infty}$ is periodic after some stage; there exist natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Choose m such that $ml \geq k$ and ml is a perfect square. Let

$$m = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_r^{\alpha_r}, \quad l = p_1^{\beta_1}p_2^{\beta_2}\ldots p_r^{\beta_r},$$

be the prime decompositions of m, l so that $\alpha_j + \beta_j$ is even for $1 \leq j \leq r$. Now take a prime p different from $p_1, p_2, \ldots, p_r$. Consider ml and pml. Since $pml - ml$ is divisible by l, we have $a_{pml} = a_{ml}$. Hence $d(pml)$ and $d(ml)$ have same parity. But $d(pml) = 2d(ml)$, since $\gcd(p, ml) = 1$ and p is a prime. Since ml is a square, $d(ml)$ is odd. It follows that $d(pml)$ is even and hence $a_{pml} \neq a_{ml}$. This contradiction implies that x is irrational.

Alternative Solution: As earlier, assume that x is rational and choose natural numbers k, l such that $a_n = a_{n+l}$ for all $n \geq k$. Consider the numbers $a_{m+1}, a_{m+2}, \ldots, a_{m+l}$, where $m \geq k$ is any number. This must contain at least one 0. Otherwise $a_n = 1$ for all $n \geq k$. But $a_r = 0$ if and only if r is a square. Hence it follows that there are no squares for $n > k$, which is absurd. Thus every l consecutive terms of the sequence $\langle a_n\rangle$ must contain a 0 after certain stage. Let $t = \max\{k, l\}$, and consider t^2 and $(t+1)^2$. Since there are no squares between t^2 and $(t+1)^2$, we conclude that $a_{t^2+j} = 1$ for $1 \leq j \leq 2t$. But then, we have $2t(> l)$ consecutive terms of the sequence $\langle a_n\rangle$ which miss 0, contradicting our earlier observation.

3. Find all real numbers x such that

$$\big[x^2 + 2x\big] = \big[x\big]^2 + 2\big[x\big].$$

(Here $[x]$ denotes the largest integer not exceeding x.)

Solution:

Adding 1 both sides, the equation reduces to

$$\big[(x+1)^2\big] = \big([x+1]\big)^2;$$

we have used $[x] + m = [x + m]$ for every integer m. Suppose $x + 1 \leq 0$. Then $[x + 1] \leq x + 1 \leq 0$. Thus

$$\big([x+1]\big)^2 \geq (x+1)^2 \geq \big[(x+1)^2\big] = \big([x+1]\big)^2.$$

Thus equality holds everywhere. This gives $[x + 1] = x + 1$ and thus $x + 1$ is an integer. Using $x + 1 \leq 0$, we conclude that

$$x \in \{-1, -2, -3, \ldots\}.$$

Suppose $x + 1 > 0$. We have

$$(x+1)^2 \geq \big[(x+1)^2\big] = \big([x+1]\big)^2.$$

Moreover, we also have

$$(x+1)^2 \leq 1 + \big[(x+1)^2\big] = 1 + \big([x+1]\big)^2.$$

Thus we obtain

$$[x]+1=[x+1]\le(x+1)<\sqrt{1+\big([x+1]\big)^2}=\sqrt{1+\big([x]+1\big)^2}.$$

This shows that

$$x\in\big[n,\sqrt{1+(n+1)^2}-1\big),$$

where $n\ge -1$ is an integer. Thus the solution set is

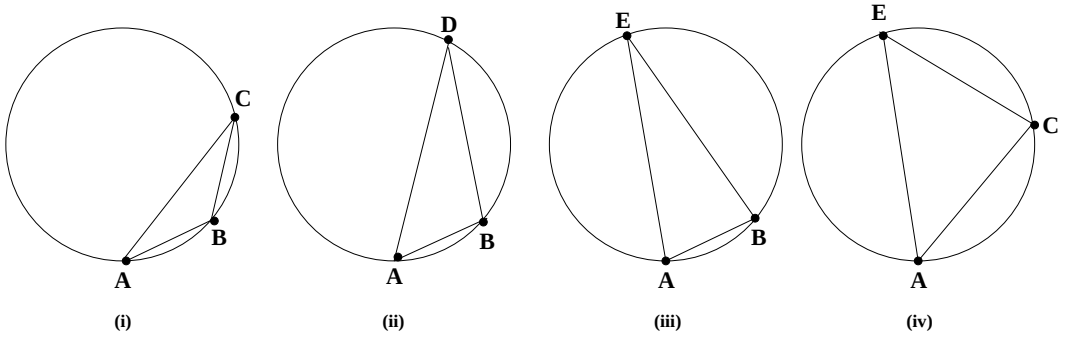
$$\{-1,-2,-3,\dots\}\cup\Big\{\cup_{n=-1}^{\infty}\,[n,\sqrt{1+(n+1)^2}-1)\Big\}.$$

It is easy verify that all the real numbers in this set indeed satisfy the given equation.

4. All the points in the plane are coloured using three colours. Prove that there exists a triangle with vertices having the same colour such that *either* it is isosceles *or* its angles are in geometric progression.

Solution:

Consider a circle of positive radius in the plane and inscribe a regular heptagon $ABCDEFG$ in it. Since the seven vertices of this heptagon are coloured by three colours, some three vertices have the same colour, by pigeon-hole principle. Consider the triangle formed by these three vertices. Let us call the part of the circumference separated by any two consecutive vertices of the heptagon an *arc*. The three vertices of the same colour are separated by *arcs* of length l,m,n as we move, say counter-clockwise, along the circle, starting from a fixed vertex among these three, where $l+m+n=7$. Since, the order of l,m,n does not matter for a triangle, there are four possibilities: 1+1+5=7; 1+2+4=7; 1+3+3=7; 2+2+3=7. In the first, third and fourth cases, we have isosceles triangles. In the second case, we have a triangle whose angles are in geometric progression. The four corresponding figures are shown below.

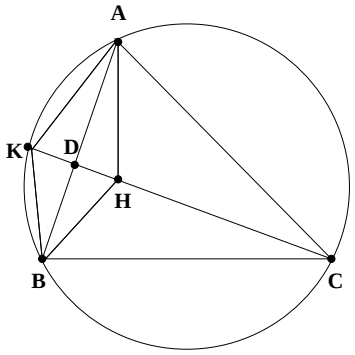


In (i), $AB=BC$; in (iii), $AE=BE$; in (iv), $AC=CE$; and in (ii) we see that $\angle D=\pi/7$, $\angle A=2\pi/7$ and $\angle B=4\pi/7$ which are in geometric progression.

5. Let ABC be an acute-angled triangle and let H be its ortho-centre. Let $h_{\max}$ denote the largest altitude of the triangle ABC. Prove that

$$AH+BH+CH\le 2h_{\max}.$$

Solution:



Let $\angle C$ be the smallest angle, so that $CA \geq AB$ and $CB \geq AB$. In this case the altitude through C is the longest one. Let the altitude through C meet AB in D and let H be the ortho-centre of ABC. Let CD extended meet the circum-circle of ABC in K. We have $CD = h_{\max}$ so that the inequality to be proved is

$$AH + BH + CH \leq 2CD.$$

Using $CD = CH + HD$, this reduces to $AH + BH \leq CD + HD$. However, we observe that $AH = AK$, $BH = BK$ and $HD = DK$.(For example $BH = BK$ and $DH = DK$ follow from the congruency of the right-angled triangles DBK and DBH.)

Thus we need to prove that $AK + BK \leq CK$. Applying Ptolemy's theorem to the cyclic quadrilateral $BCAK$, we get

$$AB \cdot CK = AC \cdot BK + BC \cdot AK \geq AB \cdot BK + AB \cdot AK.$$

This implies that $CK \geq AK + BK$, which is precisely what we are looking for.

There were other beautiful solutions given by students who participated in INMO-2009. We record them here.

1. Let AD, BE, CF be the altitudes and H be the ortho-centre. Observe that

$$\frac{AH}{AD} = \frac{[AHB]}{[ADB]} = \frac{[AHC]}{[ADC]}.$$

This gives

$$\frac{AH}{AD} = \frac{[AHB] + [AHC]}{[ADB] + [ADC]} = 1 - \frac{[BHC]}{[ABC]}.$$

Similar expressions for the ratios BH/BE and CH/CF may be obtained. Adding, we get

$$\frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

Suppose AD is the largest altitude. We get

$$\frac{AH}{AD} + \frac{BH}{AD} + \frac{CH}{AD} \leq \frac{AH}{AD} + \frac{BH}{BE} + \frac{CH}{CF} = 2.$$

This gives the result.

2. Let O be the circum-centre and let L, M, N be the mid-points of BC, CA, AB respectively. Then we know that $AH = 2OL$, $BH = 2OM$ and $CH = 2ON$. As earlier, assume AD is the largest altitude. Then BC is the least side. We have

$$\begin{aligned} 4[ABC] = 4[BOC] + 4[COA] + 4[AOB] &= BC \times 2OL + CA \times 2OM + AB \times 2ON \\ &= BC \times AH + CA \times BH + AB \times CH \\ &\geq AB(AH + BH + CH). \end{aligned}$$

Thus

$$AH + BH + CH \leq \frac{4[ABC]}{AB} = 2AD.$$

3. We make use of the fact that $AH = 2R\cos\angle A$, $BH = 2R\cos\angle B$, $CH = 2R\cos\angle C$ and $AD = 2R\sin\angle B\sin\angle C$, where R is the circum-radius of ABC. We are assuming that AD is the largest altitude so that $\angle A$ is the least angle. Thus we have to prove that

$$\cos\angle A + \cos\angle B + \cos\angle C \leq 2\sin\angle B\angle C,$$

under the assumption $\angle A \leq \angle B$ and $\angle A \leq \angle C$. On multiplying this by $2\sin\angle A$, this is equivalent to

$$2(\sin\angle A\cos\angle A + \sin\angle A\cos\angle B + \sin\angle A\cos\angle C) \leq 4\sin\angle A\sin\angle B\angle C = \sin 2A + \sin 2B + \sin 2C.$$

This is equivalent to

$$\cos\angle B(\sin\angle A - \sin\angle B) + \cos\angle C(\sin\angle A - \sin\angle C) \leq 0.$$

Since ABC is acute-angled and A is the least angle, the result follows.

6. Let a, b, c be positive real numbers such that $a^3 + b^3 = c^3$. Prove that

$$a^2 + b^2 - c^2 > 6(c-a)(c-b).$$

Solution:

The given inequality may be written in the form

$$7c^2 - 6(a+b)c - (a^2 + b^2 - 6ab) < 0.$$

Putting $x = 7c^2$, $y = -6(a+b)c$, $z = -(a^2+b^2-6ab)$, we have to prove that $x + y + z < 0$. Observe that x, y, z are not all equal($x > 0$, $y < 0$). Using the identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right],$$

we infer that it is sufficient to prove $x^3 + y^3 + z^3 - 3xyz < 0$. Substituting the values of x, y, z, we see that this is equivalent to

$$343c^6 - 216(a+b)^3c^3 - (a^2+b^2-6ab)^3 - 126c^3(a+b)(a^2+b^2-6ab) < 0.$$

Using $c^3 = a^3 + b^3$, this reduces to

$$343\left(a^3+b^3\right)^2 - 216(a+b)^3(a^3+b^3) - (a^2+b^2-6ab)^3 - 126((a^3+b^3)(a+b)(a^2+b^2-6ab) < 0.$$

This may be simplified (after some tedious calculations) to,

$$-a^2b^2(129a^2 - 254ab + 129b^2) < 0.$$

But $129a^2 - 254ab + 129b^2 = 129(a-b)^2 + 4ab > 0$. Hence the result follows.

Remark: The best constant θ in the inequality $a^2 + b^2 - c^2 \geq \theta(c-a)(c-b)$, where a, b, c

are positive reals such that $a^3+b^3=c^3$, is $\theta = 2\big(1+2^{1/3}+2^{-1/3}\big)$.

Here again, there were some beautiful solutions given by students.

1. We have

$$a^3 = c^3 - b^3 = (c-b)(c^2+cb+b^2),$$

which is same as

$$\frac{a^2}{c-b} = \frac{c^2+cb+b^2}{a}.$$

Similarly, we get

$$\frac{b^2}{c-a} = \frac{c^2+ca+a^2}{b}.$$

We observe that

$$\frac{a^2}{c-b} + \frac{b^2}{c-a} = \frac{c(a^2+b^2)-a^3-b^3}{(c-a)(c-b)} = \frac{c(a^2+b^2-c^2)}{(c-a)(c-b)}.$$

This shows that

$$\frac{a^2+b^2-c^2}{(c-a)(c-b)} = \frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb}.$$

Thus it is sufficient to prove that

$$\frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb} \geq 6.$$

However, we have $c^2+b^2 \geq 2cb$ and $c^2+a^2 \geq 2ca$. Hence

$$\frac{c^2+cb+b^2}{ca} + \frac{c^2+ca+a^2}{cb} \geq 3\left(\frac{b}{a}+\frac{a}{b}\right) \geq 3\times 2 = 6.$$

We have used AM-GM inequality.

2. Let us set $x=a/c$ and $y=b/c$. Then $x^3+y^3=1$ and the inequality to be proved is $x^2+y^2-1 > 6(1-x)(1-y)$. This reduces to

$$(x+y)^2 + 6(x+y) - 8xy - 7 > 0. \tag{1}$$

But

$$1 = x^3+y^3 = (x+y)(x^2-xy+y^2),$$

which gives $xy = \big((x+y)^3-1\big)/3(x+y)$. Substituting this in (1) and introducing $x+y=t$, the inequality takes the form

$$t^2 + 6t - \frac{8}{3}\frac{(t^3-1)}{t} - 7 > 0. \tag{2}$$

This may be simplified to $-5t^3+18t^2-2t+8>0$. Equivalently

$$-(5t-8)(t-1)^2 > 0.$$

Thus we need to prove that $5t<8$. Observe that $(x+y)^3 > x^3+y^3 = 1$, so that $t>1$. We also have

$$\left(\frac{x+y}{2}\right) \leq \frac{x^3+y^3}{2} = \frac{1}{2}.$$

This shows that $t^3 \leq 4$. Thus

$$\left(\frac{5t}{8}\right)^3 \leq \frac{125 \times 4}{512} = \frac{500}{512} < 1.$$

Hence $5t < 8$, which proves the given inequality.

3. We write $b^3 = c^3 - a^3$ and $a^3 = c^3 - b^3$ so that

$$c - a = \frac{b^3}{c^2 - ca + a^2}, \quad c - b = \frac{a^3}{c^2 - cb + b^2}.$$

Thus the inequality reduces to

$$a^2 + b^2 - c^2 > 6\frac{a^3b^3}{(c^2 - ca + a^2)(c^2 - cb + b^2)}.$$

This simplifies(after some lengthy calculations) to

$$-c^6 - (a+b)c^5 - abc^4 + (a^3+b^3)c^3 + (a^4 + a^3b + a^2b^2 + ab^3 + b^4)c^2$$
$$(a^2b + ab^2 + a^3 + b^3)abc + (a^4b^2 - 6a^3b^3 + a^2b^4) > 0.$$

Substituting

$$c^3 = a^3 + b^3, \quad c^4 = c(a^3 + b^3), \quad c^5 = c^2(a^3 + b^3), \quad c^6 = (a^3 + b^3)^2,$$

the inequality further reduces to

$$a^2b^2(a^2 + b^2 + c^2 + ac + bc - 6ab) > 0.$$

Thus we need to prove that $a^2 + b^2 + c^2 + ac + bc - 6ab > 0$. Since $a^2 + b^2 \geq 2ab$, it is enough to prove that $c^2 + c(a+b) - 4ab > 0$. Multiplying this by c and using $a^3 + b^3 = c^3$, we need to prove that

$$a^3 + b^3 + c^2a + c^2b > 4abc.$$

Using AM-GM inequality to these 4 terms and using $c > a$, $c > b$ we get

$$a^3 + b^3 + c^2a + c^2b > 4(a^3b^3c^2ac^2b)^{1/4} = 4abc,$$

which proves the inequality.